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Uniform approximation for bifurcations of periodic orbits with high repetition numbers

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Abstract. The semiclassical contribution of a periodic orbit to the quantum density of states diverges when the orbit bifurcates. In this case one has to apply approximations which are uniformly valid both in $\hbar$ and a parameter ε which describes the distance to the bifurcation. The form of the approximation depends on the repetition number m of the orbit that bifurcates. In a two-dimensional system, the approximations are different for $m = 1$ up to $m = 5$, and for $m \geqslant 5$ they have the same form as for $m = 5$. In this article, we consider the case $m \geqslant 5$ which occurs first when an integrable system is perturbed. A uniform approximation for the contribution to the spectral density is derived, which in the limit of large ε reduces to a sum of semiclassical contributions of isolated periodic orbits.

1. Introduction

Semiclassical approximations describe quantum mechanical quantities in the limit when $\hbar$ is small in comparison with relevant actions of the corresponding classical problem. For the quantum density of states $d(E)$, the semiclassical approximation is a sum of two terms, a smooth function which describes the average density of states and an oscillatory function which is a sum over contributions from the periodic orbits of the classical system. The way in which the periodic orbits contribute to this sum depends on whether they are isolated or appear in families.

If the classical system is chaotic then the periodic orbits are typically isolated and their contribution to the semiclassical level density has been derived by Gutzwiller [1, 2]. If the classical system is integrable, then the periodic orbits typically appear in tori, and these tori give a collective contribution to the level density [3, 4]. Families related to more general symmetries can also be treated [5, 6]. In general, there can also be families of orbits in the chaotic case like in the stadium or Sinai billiards, and isolated periodic orbits in the integrable case like in the ellipse billiard.

There are, however, situations where the semiclassical contributions of isolated orbits or families of orbits fail to give an accurate approximation to the level density. This is the case if, close to a periodic orbit, there exist other periodic orbits with an action difference which is not small in comparison to $\hbar$, a situation which typically occurs in a mixed system. Then there often exists a second small parameter ε besides $\hbar$ which governs the separation of the neighbouring periodic orbits. Instead of applying stationary phase approximations as

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is done in the case of isolated periodic orbits one then has to apply approximations which are uniformly valid in ε and $\hbar$.

A generic situation in mixed systems where the semiclassical approximation fails is when a bifurcation of an orbit occurs. We discuss this case by considering Gutzwiller's semiclassical contribution of an isolated stable periodic orbit with repetition number m to the level density of a two-dimensional system:

$$d_\gamma(E) = \frac{1}{\pi\hbar}\frac{T_\gamma(E)}{m}\frac{\cos\left(\frac{S_\gamma(E)}{\hbar} - \frac{\pi\nu_\gamma}{2}\right)}{\sqrt{|\mathrm{Tr}M_\gamma - 2|}} = \frac{1}{\pi\hbar}\frac{T_\gamma(E)}{m}\frac{\sin\left(\frac{S_\gamma(E)}{\hbar}\right)}{2\sin\left(\frac{\alpha_\gamma}{2}\right)}. \tag{1}$$

Here $T_\gamma(E)$, $S_\gamma(E)$, M_γ and ν_γ are the period, action, monodromy matrix and Maslov index of the orbit, respectively. The second form in equation (1) follows from the fact that the monodromy matrix M_γ of a stable periodic orbit has eigenvalues $\lambda_{1,2} = \exp(\pm \mathrm{i}\alpha_\gamma)$ and its Maslov index is given by $\nu_\gamma = 2[\frac{\alpha_\gamma}{2\pi}] + 1$. In terms of the quantities of the primitive orbit $T_\gamma(E) = mT_{\gamma,p}(E)$, $S_\gamma(E) = mS_{\gamma,p}(E)$ and $M_\gamma = (M_{\gamma,p})^m$. At a bifurcation of the orbit the angle α_γ has values $\alpha_\gamma = m\alpha_{\gamma,p} = 2\pi n$, for integer n, and expression (1) diverges. This is due to the fact that equation (1) is derived under the assumption that the orbit is isolated, which is not correct at the bifurcation. The way in which the contribution $d_\gamma(E)$ has to be modified depends on the kind of the bifurcation.

The different forms of generic bifurcations have been classified by Meyer [7] and they are discussed in [8]. They depend on the smallest repetition number of a periodic orbit for which the bifurcation occurs. If this number is m, then the bifurcation is a period m-tupling bifurcation, i.e. the primitive periods of the satellite orbits that bifurcate from the central orbit are m times the primitive period of the central orbit at the bifurcation. For m up to $m = 5$ generic bifurcations have structures that are different for every m, for example the number of orbits before and after the bifurcation are in general different. For higher repetition numbers $m > 5$ the bifurcations follow the same pattern as for $m = 5$: a stable orbit bifurcates into two satellite orbits, one stable and one unstable, and a central stable orbit.

By a transformation to normal form coordinates, Ozorio de Almeida and Hannay derived uniform approximations for the contributions of orbits near generic bifurcations [9] to the level density $d(E)$. For the cases $m \leqslant 4$ their results were given by diffraction catastrophe integrals. For the case $m \geqslant 5$ they derived a uniform approximation in terms of a J_0-Bessel function and a Fresnel integral. We will restrict ourselves in the following to $m \geqslant 5$, which is the case that occurs first when an integrable system is perturbed. The approximation of Ozorio de Almeida and Hannay for $m \geqslant 5$ is valid in the vicinity of the bifurcation, but it does not extend to the limit of contributions of isolated periodic orbits for larger values of $\varepsilon = \alpha_\gamma - 2\pi n$, i.e. further away from the bifurcation. In the present article a uniform approximation is derived which has this correct limit. The result is obtained by a variation of the method of Ozorio de Almeida and Hannay and by the inclusion of higher-order terms. The article is organized as follows. In section 2 the transformation to normal form coordinates is performed. In section 3 classical properties of the satellite orbits are derived from the normal form expansion. In section 4 a uniform approximation is applied and the contribution to the trace formula is derived, and section 5 contains a short summary.

Another example for the failing of the semiclassical approximation is the break-up of the tori of an integrable system when the system is perturbed. By applying a uniform approximation, Ozorio de Almeida showed how the torus contributions to the level density have to be modified for small perturbations [10]. For systems with a broken continuous symmetry a corresponding formula was obtained by Creagh [11]. In recent work by Tomsovic, Ullmo and Grindberg the results of Ozorio de Almeida were extended in order to obtain a formula which has the correct limits as the perturbation is increased [12, 13].

2. Transformation to normal form coordinates

The derivation starts from the semiclassical approximation to the Green function for a two-dimensional conservative system which is given by [2]

$$G(\boldsymbol{x}', \boldsymbol{x}, E) \approx \frac{1}{\mathrm{i}\hbar\sqrt{2\pi\mathrm{i}\hbar}} \sum_{\gamma} \sqrt{|D_\gamma|} \exp\left\{\frac{\mathrm{i}}{\hbar} S_\gamma(\boldsymbol{x}', \boldsymbol{x}, E) - \frac{\mathrm{i}\pi}{2}\nu_\gamma\right\} \tag{2}$$

where the sum runs over all classical trajectories from $\boldsymbol{x}$ to $\boldsymbol{x}'$ at energy E. The action S_γ along the trajectories is defined as

$$S_\gamma(\boldsymbol{x}', \boldsymbol{x}, E) = \int_\gamma \boldsymbol{p} \cdot \mathrm{d}\boldsymbol{x} \tag{3}$$

and it obeys generating function conditions relating final coordinates $(\boldsymbol{x}', \boldsymbol{p}')$ and initial coordinates $(\boldsymbol{x}, \boldsymbol{p})$

$$\frac{\partial S_\gamma}{\partial \boldsymbol{x}'} = \boldsymbol{p}' \qquad \frac{\partial S_\gamma}{\partial \boldsymbol{x}} = -\boldsymbol{p} \qquad \frac{\partial S_\gamma}{\partial E} = T. \tag{4}$$

D_γ is the determinant of a matrix of second derivatives of S_γ

$$D_\gamma = \det \begin{pmatrix} \dfrac{\partial^2 S_\gamma}{\partial \boldsymbol{x}' \partial \boldsymbol{x}} & \dfrac{\partial^2 S_\gamma}{\partial \boldsymbol{x}' \partial E} \\ \dfrac{\partial^2 S_\gamma}{\partial E \partial \boldsymbol{x}} & \dfrac{\partial^2 S_\gamma}{\partial E^2} \end{pmatrix} \tag{5}$$

and ν_γ is the number of conjugate points along the trajectory γ (for fixed energy).

The spectral density is obtained from the Green function by

$$d(E) = \sum_n \delta(E - E_n) = -\frac{1}{\pi} \,\mathrm{Im} \int \mathrm{d}^2x'\, \mathrm{d}^2x\, \delta(\boldsymbol{x}' - \boldsymbol{x}) G(\boldsymbol{x}', \boldsymbol{x}, E). \tag{6}$$

In the derivation of Gutzwiller's trace formula two of the integrals are evaluated by an integration over the δ-function. The remaining two integrals are evaluated in the vicinity of periodic orbits by choosing local coordinates $\boldsymbol{x} = (z, y)$, where z is the coordinate along a periodic orbit and y is the coordinate perpendicular to it. The integral over y is evaluated in stationary phase approximation and the integral over z is then an integral over a constant. In this way one obtains a sum of contributions of isolated periodic orbits.

We now consider the contribution of a bifurcating orbit with repetition number m to the level density. Also in this case the local coordinates z and y are introduced. The variable z is a periodic coordinate with period l_γ/m where l_γ is the length of the orbit. The conditions $E =$ constant and $z =$ constant define local Poincaré surfaces of section in phase space, and the action S_γ is the generating function for the mth iterate of the Poincaré map:

$$\frac{\partial S_\gamma}{\partial y'} = p_y' \qquad \frac{\partial S_\gamma}{\partial y} = -p_y. \tag{7}$$

Near the bifurcation the mth iterate of the Poincaré map is close to the identity [8]. The identity transformation, however, cannot be generated by a generating function which depends on final and initial y-coordinates. Instead one has to use a mixed coordinate–momentum representation. This can be done by starting with the Green function which is locally represented in a coordinate–momentum representation. One arrives at the same result, if one replaces one of the delta-functions in (6) by

$$\delta(y'-y) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} \mathrm{d}p_y \, \mathrm{e}^{\frac{\mathrm{i}}{\hbar} p_y (y-y')} \tag{8}$$

and evaluates the integral over y in stationary phase approximation. Then the contribution from the vicinity of the orbit to the level density is given by

$$d_\gamma(E) \approx 2\,\mathrm{Re}\int \mathrm{d}z\,\mathrm{d}y'\,\mathrm{d}p_y \,\frac{\sqrt{|\tilde{D}_\gamma|}}{(2\pi\hbar)^2}\exp\left\{\frac{\mathrm{i}}{\hbar}\tilde{S}_\gamma(z',y',z,p_y,E) - \frac{\mathrm{i}}{\hbar}y'p_y - \frac{\mathrm{i}\pi}{2}\tilde{\nu}_\gamma\right\}\Bigg|_{z'=z+l_\gamma} \tag{9}$$

where the new generating function $\tilde{S}_\gamma$ is the Legendre transform of S_γ

$$\tilde{S}_\gamma(z',y',z,p_y,E) = S_\gamma(z',y',z,y,E) + yp_y \tag{10}$$

and the value of y is taken at the stationary point $\partial S_\gamma/\partial y + p_y = 0$. $\tilde{D}$ has the same form as D in (5), with the replacements $S_\gamma \to \tilde{S}_\gamma$, $\boldsymbol{x}' \to (z', y')$ and $\boldsymbol{x} \to (z, p_y)$.

The exponent in (9), when considered as a function of y' and p_y, has stationary points at the central orbit as well as at the satellite orbits. A stationary phase approximation would yield a sum of semiclassical contributions of these orbits which diverge at the bifurcation.

In order to obtain a non-divergent uniform approximation for the joint contribution of these orbits one has to expand the action around the central orbit in higher order. In general, this results in a complicated expansion in the variables z, y' and p_y. The integrals can be considerably simplified by a canonical transformation of the coordinates and by using the fact that the form of equation (9) is semiclassically invariant under canonical transformations, i.e. if one replaces the old coordinates in (9) by new canonical coordinates

$$(z, p_z, y, p_y) \to (Z, P_Z, Y, P_Y) \tag{11}$$

and the generating function $\tilde{S}_\gamma$ by the generating function for the new coordinates, then a stationary phase approximation for the integrals gives the same result as before. This follows from work of Miller [14] and is discussed by Littlejohn [15]. There is a restriction to the above statement. It is true as long as there are no bounds on the new coordinates. If the new coordinates are bounded then there are modifications as is discussed in the following.

The most simple form that the generating function can take near the bifurcation is given by the normal form. Ozorio de Almeida and Hannay performed the transformation to the normal form in order to obtain the contribution of orbits close to a bifurcation [9]. The subject of the present work is to obtain an approximation which has the right limit as one moves away from the bifurcation. For that purpose, higher-order terms have to be included in the expansion. The transformation is described in appendix A and it results in the replacement

$$\tilde{S}_\gamma(z',y',z,p_y,E) - p_y y' \to \hat{S}_\gamma(z',\Phi',z,I,E) - I\Phi' \tag{12}$$

where $\hat{S}_\gamma$ is of the form

$$\hat{S}_\gamma(z',\Phi',z,I,E)|_{z'=z+l_\gamma} = S_0(E) + S(I,\Phi',E). \tag{13}$$

$S_0(E)$ is the action along the central orbit and the expansion of the generating function S for mth iterate of the Poincaré map with $m \geqslant 5$ is given by

$$S(I,\Phi',E) = I\Phi' - \varepsilon I - \sum_{\nu=2}^{m-2} c_\nu I^\nu - \sum_{\nu=0}^{[\frac{m}{2}]-2} a_\nu I^{\frac{m}{2}+\nu}\sin(m\Phi'). \tag{14}$$

Here $\varepsilon = \alpha_0 - 2\pi n = m\alpha_{0,p} - 2\pi n$ is a parameter which is zero at the bifurcation, and the index 0 denotes quantities of the central periodic orbit. I and Φ are canonical polar coordinates. The expansion is carried out up to order I^{m-2} since in all lower-order approximations the two satellite orbits have the same semiclassical amplitude as will be shown in the next section.

It is important to note that the transformation to the I- and Φ-coordinates is a semiclassical transformation since the quantization of the angle coordinate Φ leads to difficulties. Due to the periodicity of the angle Φ, the quantum mechanical operator corresponding to I has a discrete spectrum. From its form $I = (P_Y^2 + Y^2)/2$ in terms of unbounded canonical coordinates Y and P_Y (see appendix A) it follows that its spectrum is that of the harmonic oscillator $I_n = (n + 1/2)\hbar$, for $0 \leqslant n < \infty$. On the other hand, it is not possible to define a Hermitian operator corresponding to the phase Φ since this leads to contradictions [16]. In the semiclassical approximation, however, it is still possible to work with I and Φ coordinates if one restricts integrations over Φ to a range of 2π and replaces integrations over I by

$$\int \mathrm{d}I \to \hbar \sum_{I_n} \tag{15}$$

as discussed in [17]. In this way one obtains the following expression for the contribution to the spectral density

$$d_\gamma(E) \approx 2\,\mathrm{Re} \int_0^{l_\gamma/m} \mathrm{d}z \int_0^{2\pi} \mathrm{d}\Phi' \sum_{n=0}^{\infty} \frac{\hbar}{(2\pi\hbar)^2} \sqrt{|\hat{D}_\gamma|} \times \exp\left\{\frac{\mathrm{i}}{\hbar} S_0(E) + \frac{\mathrm{i}}{\hbar} S(I_n, \Phi', E) - \frac{\mathrm{i}}{\hbar} I_n \Phi' - \frac{i\pi}{2}\nu\right\}. \tag{16}$$

As a consequence of the transformation to normal form coordinates the integrand of the integrals in (16) no longer depends on z, and the z-integration can now be done trivially. Furthermore, the determinant $\hat{D}_\gamma$ can be reduced to the following form (see discussion in [15]):

$$\hat{D}_\gamma = \det \begin{pmatrix} \frac{\partial^2 \hat{S}_\gamma}{\partial z' \partial z} & \frac{\partial^2 \hat{S}_\gamma}{\partial z' \partial I} & \frac{\partial^2 \hat{S}_\gamma}{\partial z' \partial E} \\ \frac{\partial^2 \hat{S}_\gamma}{\partial \Phi' \partial z} & \frac{\partial^2 \hat{S}_\gamma}{\partial \Phi' \partial I} & \frac{\partial^2 \hat{S}_\gamma}{\partial \Phi' \partial E} \\ \frac{\partial^2 \hat{S}_\gamma}{\partial E \partial z} & \frac{\partial^2 \hat{S}_\gamma}{\partial E \partial I} & \frac{\partial^2 \hat{S}_\gamma}{\partial E \partial E} \end{pmatrix} = \frac{1}{\dot{z}'\dot{z}} \frac{\partial^2 S}{\partial \Phi' \partial I}. \tag{17}$$

Here $\dot{z}$ and $\dot{z}'$ are the velocities in the transformed system which are in general not identical to those of the original system. For a periodic orbit $\dot{z} = \dot{z}'$ and

$$\int_0^{l_\gamma/m} \mathrm{d}z \sqrt{\frac{1}{\dot{z}'\dot{z}}} = \frac{1}{m} \frac{\partial \hat{S}_\gamma}{\partial E} = \frac{T(\Phi', I, E)}{m} \tag{18}$$

where T is the time from z to z' along the orbit. This is also true for non-periodic orbits within the approximations for the exponential prefactor that will be discussed in the next section and in appendix C.

After integration over z and an application of the Poisson summation formula for the sum over n one obtains

$$d_\gamma(E) \approx 2\,\mathrm{Re} \sum_{L=-\infty}^{\infty} \int_0^{2\pi} \mathrm{d}\Phi' \int_0^{\infty} \mathrm{d}I \; \frac{1}{(2\pi\hbar)^2} \frac{1}{m} \frac{\partial \hat{S}_\gamma}{\partial E} \left| \frac{\partial^2 S}{\partial I \, \partial \Phi'} \right|^{\frac{1}{2}} \times \exp\left\{ \frac{\mathrm{i}}{\hbar} S_0(E) + \frac{\mathrm{i}}{\hbar} S(I, \Phi', E) - \frac{\mathrm{i}}{\hbar} I \Phi' - \frac{\mathrm{i}\pi}{2} \nu + 2\pi \mathrm{i} \left(\frac{I}{\hbar} - \frac{1}{2} \right) L \right\}. \qquad (19)$$

Due to the coordinate transformation to polar canonical coordinates the form of the semiclassical approximation has changed in comparison with (9). The origin of the coordinate system is not a stationary point any more. Instead, the Gutzwiller contribution of the central orbit can now be obtained by a summation over L of the contributions from the boundary $I = 0$ of the I-integration. The semiclassical contributions of the satellite orbits are contained in the $L = 0$ term. All other terms $L \neq 0$ have no stationary point near the origin, and they contribute semiclassically only with a boundary contribution.

3. Properties of the satellite orbits

Before continuing with the further evaluation of the integrals we consider properties of the satellite orbits that can be obtained from the normal form expansion of the generating function (14).

The stationary points of the exponent in (19) for $L = 0$ are determined by

$$\begin{aligned} I &= \frac{\partial S}{\partial \Phi'} = I - \sum_{\nu=0}^{[\frac{m}{2}]-2} a_\nu m I^{\frac{m}{2}+\nu} \cos(m\Phi') \\ \Phi' &= \frac{\partial S}{\partial I} = \Phi' - \varepsilon - \sum_{\nu=2}^{m-2} \nu c_\nu I^{\nu-1} - \sum_{\nu=0}^{[\frac{m}{2}]-2} a_\nu \left(\frac{m}{2} + \nu \right) I^{\frac{m}{2}+\nu-1} \sin(m\Phi'). \end{aligned} \qquad (20)$$

The first equation is solved by $\cos(m\Phi') = 0$, and the second equation then determines the value of I at the stationary points. There are altogether $2m$ solutions of the equations (20); m solutions with $\sin(m\Phi') = 1$ correspond to the satellite orbit which is labelled by 1 in the following, and m solutions with $\sin(m\Phi') = -1$ correspond to the satellite orbit labelled by 2.

It will be convenient for the following calculations to solve the second equation in (20) for I as a function of Φ' and insert the values of Φ' at the stationary points at the end. The solution can be expressed as a sum over powers of ε up to order ε^{m-3}:

$$I^*(\Phi') = \sum_{\nu=2}^{m-2} c'_\nu \hat{\varepsilon}^{\nu-1} + \sum_{\nu=0}^{[\frac{m}{2}]-2} a'_\nu \hat{\varepsilon}^{\frac{m}{2}+\nu-1} \sin(m\Phi') + b'_0 \hat{\varepsilon}^{m-3} \cos^2(m\Phi') \qquad (21)$$

where the definition for $\hat{\varepsilon}$ and the first coefficients in the three sums are given by

$$\hat{\varepsilon} = -\frac{\varepsilon}{2c_2} \qquad c'_2 = 1 \qquad a'_0 = -\frac{a_0 m}{4c_2} \qquad b'_0 = -\frac{a_0^2 m^2}{16 c_2^2} \left(\frac{m}{2} - 1 \right). \qquad (22)$$

From the leading term in (21) it follows that I^* is positive if $\hat{\varepsilon} > 0$, i.e. the satellite orbits are real for positive $\hat{\varepsilon}$. Figure 1 shows the stationary points of $S(I, \Phi', E) - I\Phi'$ for $m = 5$ together with a contour line. The dots correspond to the stable periodic orbits and the crosses to the unstable periodic orbit. The satellite orbits are each represented by five stationary points since they cross the Poincaré section of surface five times.

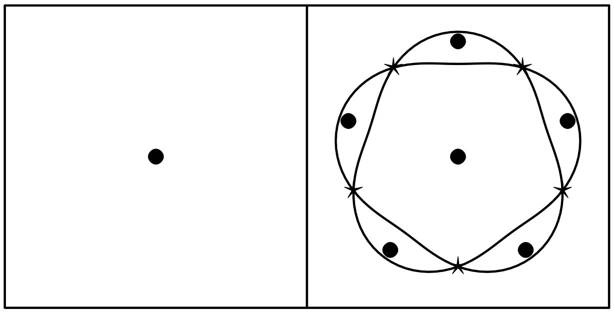

Figure 1. The positions of the periodic orbits in the Poincaré section of surface for $\hat{\varepsilon} < 0$ (left) and $\hat{\varepsilon} > 0$ (right), and a contour line of $S(I, \Phi', E) - I\Phi'$.

Inserting the value $I = I^*$ into the function $S(I, \Phi', E)$ one obtains

$$S(I^*, \Phi', E) = I^*\Phi' + \sum_{\nu=2}^{m-2} c''_\nu \hat{\varepsilon}^\nu + \sum_{\nu=0}^{[\frac{m}{2}]-2} a''_\nu \hat{\varepsilon}^{\frac{m}{2}+\nu} \sin(m\Phi') + b''_0 \hat{\varepsilon}^{m-2} \cos^2(m\Phi') \tag{23}$$

where the first coefficients in the sums are given by

$$c''_2 = c_2 \qquad a''_0 = -a_0 \qquad b''_0 = -\frac{a_0^2 m^2}{16 c_2}. \tag{24}$$

From (23) the expansions of the actions of the satellite orbits are obtained:

$$\begin{aligned} S_{1,2}(E) &= [S_0(E) + S(I^*, \Phi', E) - I^*\Phi']_{\sin(m\Phi')=\pm 1} \\ &= \bar{S}(E) \pm \Delta S(E) \end{aligned} \tag{25}$$

where

$$\begin{aligned} \bar{S}(E) &= \frac{S_1(E) + S_2(E)}{2} = S_0(E) + \sum_{\nu=2}^{m-2} c''_\nu \hat{\varepsilon}^\nu \\ \Delta S(E) &= \frac{S_1(E) - S_2(E)}{2} = \sum_{\nu=0}^{[\frac{m}{2}]-2} a''_\nu \hat{\varepsilon}^{\frac{m}{2}+\nu}. \end{aligned} \tag{26}$$

It follows that $S_1(E) - S_0(E)$ and $S_2(E) - S_0(E)$ are both of order $\hat{\varepsilon}^2$ whereas $S_1(E) - S_2(E)$ is of order $\hat{\varepsilon}^{\frac{m}{2}}$. As $\hat{\varepsilon}$ is increased, the differences of the actions of the satellite orbits $S_1(E)$ and $S_2(E)$ thus increase more slowly than the difference between either one of them and $S_0(E)$. In the following we assume without loss of generality that $a_0 < 0$ since a rotation of the coordinate system by π/m changes the sign of a_0. Then $\Delta S(E) > 0$ and $S_1(E) > S_2(E)$ for positive $\hat{\varepsilon}$.

Here we also give the expansions of several derivatives of $S(I, \Phi', E)$ for $I = I^*$:

$$
\begin{aligned}
\left.\frac{\partial S}{\partial E}\right|_{I=I^*} &= \sum_{\nu=2}^{m-2} c_\nu^{(3)} \hat{\varepsilon}^{\nu-1} + \sum_{\nu=0}^{[\frac{m}{2}]-2} a_\nu^{(3)} \hat{\varepsilon}^{\frac{m}{2}+\nu-1} \sin(m\Phi') + b_0^{(3)} \hat{\varepsilon}^{m-3} \cos^2(m\Phi') \\
\left.\frac{\partial^2 S}{\partial I^2}\right|_{I=I^*} &= \sum_{\nu=2}^{m-2} c_\nu^{(4)} \hat{\varepsilon}^{\nu-2} + \sum_{\nu=0}^{[\frac{m}{2}]-2} a_\nu^{(4)} \hat{\varepsilon}^{\frac{m}{2}+\nu-2} \sin(m\Phi') + b_0^{(4)} \hat{\varepsilon}^{m-4} \cos^2(m\Phi') \\
\left.\frac{\partial^2 S}{\partial I \partial \Phi'}\right|_{I=I^*} &= 1 + \sum_{\nu=0}^{[\frac{m}{2}]-2} a_\nu^{(5)} \hat{\varepsilon}^{\frac{m}{2}+\nu-1} \cos(m\Phi') + b_0^{(5)} \hat{\varepsilon}^{m-3} \sin(m\Phi) \cos(m\Phi') \\
\left.\frac{\partial^2 S}{\partial \Phi'^2}\right|_{I=I^*} &= \sum_{\nu=0}^{[\frac{m}{2}]-2} a_\nu^{(6)} \hat{\varepsilon}^{\frac{m}{2}+\nu} \sin(m\Phi') + b_0^{(6)} \hat{\varepsilon}^{m-2} \sin^2(m\Phi')
\end{aligned}
\tag{27}
$$

and the first coefficients in the sums are given by

$$
\begin{aligned}
& c_2^{(3)} = -\frac{\partial \alpha_0}{\partial E} \qquad a_0^{(3)} = \frac{a_0 m}{4c_2} \frac{\partial \alpha_0}{\partial E} \qquad b_0^{(3)} = \frac{a_0^2 m^2}{16 c_2^2} \left(\frac{m}{2} - 1\right) \frac{\partial \alpha_0}{\partial E} \\
& c_2^{(4)} = -2c_2 \qquad a_0^{(4)} = -\frac{a_0 m}{2} \left(\frac{m}{2} - 1\right) \qquad b_0^{(4)} = -\frac{a_0^2 m^2}{8 c_2} \left(\frac{m}{2} - 1\right) \left(\frac{m}{2} - 2\right) \\
& a_0^{(5)} = -\frac{a_0 m^2}{2} \qquad b_0^{(5)} = \frac{a_0^2 m^3}{8 c_2} \left(\frac{m}{2} - 1\right) \\
& a_0^{(6)} = a_0 m^2 \qquad b_0^{(6)} = -\frac{a_0^2 m^4}{8 c_2}.
\end{aligned}
\tag{28}
$$

From the expansions in (27) the traces of the monodromy matrices of the satellite orbits are obtained as

$$
\begin{aligned}
\mathrm{Tr}\, M_{1,2} &= \left[\left(\frac{\partial^2 S}{\partial I \partial \Phi'} \right)^{-1} \left(1 + \frac{\partial^2 S}{\partial I \partial \Phi'} \frac{\partial^2 S}{\partial I \partial \Phi'} - \frac{\partial^2 S}{\partial I^2} \frac{\partial^2 S}{\partial \Phi'^2} \right) \right]_{I=I^*, \sin(m\Phi')=\pm 1} \\
&= \left[2 - \frac{\partial^2 S}{\partial I^2} \frac{\partial^2 S}{\partial \Phi'^2} \right]_{I=I^*, \sin(m\Phi')=\pm 1} \\
&= 2 + \left[\sum_{\nu=0}^{[\frac{m}{2}]-2} a_\nu^{(7)} \hat{\varepsilon}^{\frac{m}{2}+\nu} \sin(m\Phi') + b_0^{(7)} \hat{\varepsilon}^{m-2} \sin^2(m\Phi') \right]_{\sin(m\Phi')=\pm 1}
\end{aligned}
\tag{29}
$$

where

$$
a_0^{(7)} = 2 a_0 c_2 m^2 \qquad b_0^{(7)} = -\frac{a_0^2 m^3}{2}. \tag{30}
$$

For both satellite orbits $(\mathrm{Tr}\, M - 2)$ is of order order $\hat{\varepsilon}^{\frac{m}{2}}$ and it is determined only up to a relative order of $\hat{\varepsilon}^{\frac{m}{2}-2}$ with respect to the leading term. For that reason, the amplitude in the semiclassical approximation (which is a limit of the present uniform approximation) is only valid within this relative order of $\hat{\varepsilon}$. In all following calculations it is therefore sufficient to expand the exponential prefactor in all integrals only up to order $\hat{\varepsilon}^{\frac{m}{2}-2}$.

The difference in the values of $|\,\mathrm{Tr}\, M - 2|$ for the two satellite orbits is of order $\hat{\varepsilon}^{m-2}$. This is why the expansion in (14) had to be carried out up to order I^{m-2} since in a lower order the two satellite orbits have the same semiclassical amplitude.

From the expansions of $\mathrm{Tr}\, M_{1,2}$ and $S_{1,2}(E)$ one can obtain relations between the stabilities and actions of the satellite orbits. For example,

$$\lim_{\varepsilon\to 0} \frac{(\mathrm{Tr}\, M_{1,2} - 2)}{\varepsilon^{\frac{m}{2}}} = \mp \frac{m^2}{2} \lim_{\varepsilon\to 0} \frac{\Delta S(E)}{\varepsilon^{\frac{m}{2}-2}[\bar{S}(E) - S_0(E)]} \tag{31}$$

and

$$\lim_{\varepsilon\to 0} \frac{(\mathrm{Tr}\, M_1 + \mathrm{Tr}\, M_2 - 4)}{\varepsilon^{m-2}} = -\frac{m^3}{4} \lim_{\varepsilon\to 0} \frac{[\Delta S(E)]^2}{\varepsilon^{m-4}[\bar{S}(E) - S_0(E)]^2}. \tag{32}$$

Note that equations (31) and (32) are expressed in terms of ε and not $\hat{\varepsilon}$.

4. The contribution to the level density

We continue now with the further evaluation of the integrals in (19). For $L = 0$, the integral over I has a stationary point near the boundary $I = 0$ of the integral, and for that reason it cannot be evaluated by a stationary phase approximation. Berry and Tabor derived a uniform approximation for this case [3] which can be written in the form [18]

$$\begin{aligned}\int_0^\infty \mathrm{d}I\, g(I) \mathrm{e}^{\frac{\mathrm{i}}{\hbar} f(I)} = \frac{g(I^*)\sqrt{2\pi \mathrm{i}\beta\hbar}}{\sqrt{|f''(I^*)|}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar} f(I^*)} \left[\Theta(I^*) + \sqrt{\frac{i\beta}{2\pi}}\, \mathrm{sign}(I^*) \int_\Lambda^\infty \mathrm{d}X\, \frac{1}{X^2} \mathrm{e}^{\frac{\mathrm{i}\beta}{2} X^2}\right] \\ -\frac{\hbar}{\mathrm{i}} \frac{g(0)}{f'(0)} \mathrm{e}^{\frac{\mathrm{i}}{\hbar} f(0)}\end{aligned} \tag{33}$$

where I^* is determined by $f'(I^*) = 0$, $\beta = \mathrm{sign}(f''(I^*))$ and $\Lambda = \sqrt{\frac{2}{\beta\hbar}(f(0) - f(I^*))}$. $\Theta(I)$ denotes the Heaviside theta function. The terms on the right-hand side of (33) have the following interpretation: the term multiplying the Θ-function is the stationary phase approximation of the integral, the last term is the contribution from the boundary (which can be obtained by an integration by parts), and the remaining term is an interference term between the two.

The approximation (33) is applied to the I-integral for $L = 0$. For $L \neq 0$ the I-integrals give only a boundary contribution which is given by the last term on the right-hand side of (33). The result is

$$\begin{aligned} d_\gamma(E) \approx 2\,\mathrm{Re} \int_0^{2\pi} \mathrm{d}\Phi'\, \frac{\sqrt{2\pi \mathrm{i}\beta\hbar}}{(2\pi\hbar)^2} \frac{1}{m} \frac{\partial \hat{S}_\gamma}{\partial E} \frac{\left|\frac{\partial^2 S}{\partial I \partial \Phi'}\right|^{\frac{1}{2}}}{\left|\frac{\partial^2 S}{\partial I^2}\right|^{\frac{1}{2}}} \\ \times \exp\left\{\frac{\mathrm{i}}{\hbar}\bar{S}(E) + \frac{\mathrm{i}}{\hbar}\Delta S(E) \sin(m\Phi') + \frac{\mathrm{i}}{\hbar} b_0'' \hat{\varepsilon}^{m-2} \cos^2(m\Phi') - \frac{\mathrm{i}\pi}{2}\nu\right\} \\ \times \left[\Theta(\hat{\varepsilon}) + \sqrt{\frac{\mathrm{i}\beta}{2\pi}}\, \mathrm{sign}(\hat{\varepsilon}) \int_{\Lambda'}^\infty \mathrm{d}X'\, \frac{1}{X'^2} \mathrm{e}^{\frac{\mathrm{i}\beta}{2} X'^2}\right] + d_0(E) \end{aligned} \tag{34}$$

where $\beta = -\,\mathrm{sign}(c_2)$ and $\Lambda' = \sqrt{\frac{2}{\beta\hbar}(I^*\Phi' - S(I^*, \Phi', E))}$. The term $d_0(E)$ is the sum of the boundary contributions and it is identical to the semiclassical contribution of the central stable periodic orbit:

$$\begin{aligned} d_0(E) &= 2\mathrm{Re} \sum_{L=-\infty}^{\infty} \frac{2\pi}{(2\pi\hbar)^2} \frac{T_0(E)}{m} \frac{(-1)^L}{\frac{\mathrm{i}}{\hbar}\varepsilon - \frac{2\pi \mathrm{i} L}{\hbar}} \exp\left\{\frac{\mathrm{i}}{\hbar} S_0(E) - \frac{\mathrm{i}\pi}{2}\nu\right\} \\ &= \frac{1}{\pi\hbar} \frac{T_0(E)}{m} \frac{\sin\{\frac{S_0(E)}{\hbar} - \frac{\pi}{2}\nu + \pi n\}}{2 \sin\frac{\alpha_0}{2}} \end{aligned} \tag{35}$$

where the relation

$$\frac{1}{\sin(z)} = \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{z - \pi k} \tag{36}$$

has been used [19]. A comparison of (35) with (1) shows that $\nu = 2n$ (modulo 4).

The remaining integral over Θ' can be performed after a change of the angle variable. The details are given in appendix B, and the integral results in

$$\begin{aligned} d_\gamma(E) \approx{}& \Theta(\hat{\varepsilon}) \frac{\bar{A}}{\pi\hbar} \sqrt{\frac{2\pi\,\Delta S(E)}{\hbar}}\, J_0\left(\frac{\Delta S(E)}{\hbar}\right) \cos\left(\frac{\bar{S}(E)}{\hbar} - \frac{\pi(\nu_1+\nu_2)}{4}\right) \\ &+ \Theta(\hat{\varepsilon}) \frac{\Delta A}{\pi\hbar} \sqrt{\frac{2\pi\,\Delta S(E)}{\hbar}}\, J_1\left(\frac{\Delta S(E)}{\hbar}\right) \cos\left(\frac{\bar{S}(E)}{\hbar} - \frac{\pi(\nu_1+\nu_2-2)}{4}\right) \\ &- \operatorname{sign}(\hat{\varepsilon}) \frac{\bar{A}}{\pi\hbar} \sqrt{\frac{|\Delta S(E)|}{\hbar}} \int_\Lambda^\infty \mathrm{d}X\, \frac{\beta(-1)^n}{X^2} \sin\left(\frac{\bar{S}(E)}{\hbar} + \frac{\beta}{2} X^2\right) \\ &+ \frac{1}{\pi\hbar} \frac{T_0(E)}{2m \sin(\frac{\alpha_0}{2})} \sin\left(\frac{S_0(E)}{\hbar}\right) \end{aligned} \tag{37}$$

where $\nu_1 = 2n - (\beta-1)/2$, $\nu_2 = 2n - (\beta+1)/2$, $\Lambda = \sqrt{\frac{2}{\beta\hbar}[S_0(E) - \bar{S}(E)]}$, and

$$\begin{aligned} \bar{A} &= \frac{1}{2}\left[\frac{T_1(E)}{\sqrt{|\operatorname{Tr} M_1 - 2|}} + \frac{T_2(E)}{\sqrt{|\operatorname{Tr} M_2 - 2|}}\right] \\ \Delta A &= \frac{1}{2}\left[\frac{T_1(E)}{\sqrt{|\operatorname{Tr} M_1 - 2|}} - \frac{T_2(E)}{\sqrt{|\operatorname{Tr} M_2 - 2|}}\right]. \end{aligned} \tag{38}$$

This is the final result of this paper. It is an approximation which is expressed completely in terms of the periods, actions, stabilities and Maslov indices of the three periodic orbits which are involved in the bifurcation. The first two terms on the right-hand side are the joint contributions of the two satellite orbits. The third term is an interference term between the central orbit and the satellite orbits and the last term is the Gutzwiller contribution of the stable central orbit. In billiard systems there also exists an alternative form of (37) as discussed in appendix D.

In comparison with the result of Ozorio de Almeida and Hannay, which is valid near the bifurcation, equation (37) contains the following extensions: the J_1-term which is obtained by including higher-order terms in the normal form expansion, the full Gutzwiller contribution of the central orbit which is obtained by summing also over the $L \neq 0$ terms, and the periods of the satellite orbits which are obtained by expanding the whole exponential prefactor in powers of $\hat{\varepsilon}$. A further difference is that in the formula of Ozorio de Almeida and Hannay the interference term is multiplied by a J_0-Bessel function. This J_0-term was obtained by an evaluation of the Φ'-integral and a subsequent evaluation of the I-integral in which the J_0-term was considered as part of the exponential prefactor, an approximation which is valid only near the bifurcation.

As discussed in appendix B, the definition of $\bar{A}$ and ΔA in terms of the periods $T_1(E)$ and $T_2(E)$ was somewhat ambiguous since both periods are identical in the present approximation. In order to see that the dependence on the periods is correct, one has to expand the action in one order higher (for scale-invariant systems one has to include even two orders more since $\mathrm{d}\alpha_0/\mathrm{d}E = 0$). In appendix C the derivation is done by including one order more for the case $m \geqslant 7$. One then arrives exactly at the same result (37) with the previous definitions of $\bar{A}$ and ΔA. For the cases $m = 5$ and $m = 6$ the calculations are more elaborate and are not done here.

Formula (37) can also be applied to repetitions of the primitive periodic orbit with repetition numbers m' which are a multiple of m, as long as these repetitions do not undergo further bifurcations. The only difference is that in the definitions of $\bar{A}$ and ΔA, the periods T_1 and T_2 have to be replaced by the primitive periods of the satellite orbits, and in (37) m has to be replaced by m' and n by nm'/m.

We now discuss different limits of the formula (37), first for $\hat{\varepsilon} > 0$. If $\Delta S/\hbar \gg 1$ and $|S_0 - \bar{S}|/\hbar \gg 1$, i.e. if one is sufficiently far away from the bifurcation, then the Bessel functions can be replaced by their leading asymptotic terms

$$J_0(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{4}\right) \qquad J_1(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{3\pi}{4}\right) \qquad z \to \infty \tag{39}$$

and the interference term is of order $\hbar$ smaller than the other terms and can be neglected. Then $d_\gamma(E)$ is a sum of semiclassical contributions of the three isolated orbits

$$d_\gamma(E) \approx \frac{1}{\pi\hbar} \frac{T_0(E)}{2m\sin(\frac{\alpha_0}{2})} \sin\left(\frac{S_0(E)}{\hbar}\right) + \sum_{i=1}^{2} \frac{T_i(E)}{\pi\hbar\sqrt{|\operatorname{Tr} M_i - 2|}} \cos\left(\frac{S_i(E)}{\hbar} - \frac{\pi\nu_i}{2}\right). \tag{40}$$

In the other limit when $\Delta S/\hbar \ll 1$, i.e. near the bifurcation, the Bessel functions can be replaced by their value at zero argument. This yields

$$\begin{aligned} d_\gamma(E) \approx \Theta(\hat{\varepsilon}) \frac{\bar{A}}{\pi\hbar} \sqrt{\frac{2\pi\,\Delta S(E)}{\hbar}} \cos\left(\frac{\bar{S}(E)}{\hbar} - \pi n + \frac{\pi\beta}{4}\right) \\ - \operatorname{sign}(\hat{\varepsilon}) \frac{\bar{A}}{\pi\hbar} \sqrt{\frac{|\Delta S(E)|}{\hbar}} \int_\Lambda^\infty \mathrm{d}X\, \frac{\beta(-1)^n}{X^2} \sin\left(\frac{\bar{S}(E)}{\hbar} + \frac{\beta}{2} X^2\right) \\ + \frac{1}{\pi\hbar} \frac{T_0(E)}{2m\sin(\frac{\alpha_0}{2})} \sin\left(\frac{S_0(E)}{\hbar}\right). \end{aligned} \tag{41}$$

This is exactly the form that is obtained for an integrable system when a new torus arises through a stable orbit of the system [18], i.e. it is the contribution of a torus, a stable orbit and an interference term between both. The two satellites thus contribute near the bifurcation to the level density as if they would form a torus [9]. This is a consequence of the fact that $\Delta S(E)$ is of lower order in $\hat{\varepsilon}$ than $\bar{S}(E) - S_0(E)$, i.e. the values of the actions of the satellite orbits separate more slowly from each other than each of them separates from the action of the central orbit.

In the limit $\hat{\varepsilon} \to 0$ the contributions of the central orbit and the interference term diverge, but their sum remains finite. This can be seen by integrating the interference term by part which yields two terms. One of them cancels the divergence of the central orbit (this can be seen by using (31)), and the other term gives minus one-half the torus contribution. As a consequence, the whole approximation in (37) is continuous at $\varepsilon = 0$, and the value of $d_\gamma(E)$ for $\varepsilon = 0$ is given by

$$d_\gamma(E) = \frac{T_0(E)}{m} \frac{1}{\sqrt{4\pi\hbar^3|c_2|}} \cos\left(\frac{S_0(E)}{\hbar} - \pi n + \frac{\pi\beta}{4}\right). \tag{42}$$

For negative values of $\hat{\varepsilon}$ the satellite orbits are complex and contribute only through the interference term. In contrast to the case $\hat{\varepsilon} > 0$ their contribution does not split as $-\hat{\varepsilon}$ is increased. Their contribution is like that of a complex torus.

5. Summary

In contrast to unstable periodic orbits, stable orbits cannot be considered as being isolated. If the stability of the orbit changes by an arbitrarily small but finite amount, for example by changing the energy or an external parameter, then the orbit bifurcates infinitely many times. This is the case, since in a generic situation a bifurcation occurs every time when the stability angle $\alpha_{\gamma,p}$ is a rational multiple of 2π.

As a consequence, Gutzwiller's approximation for the contribution of a stable orbit to the quantum density of states fails when summed over all repetition numbers, since it was derived under the assumption that the orbit was isolated. A remedy to this situation lies in the fact that different bifurcations are separated by the different repetitions of the primitive periodic orbit. If $\alpha_{\gamma,p} = 2\pi \frac{n}{m}$, where n and m are relatively prime, then the Gutzwiller approximation diverges for the mth repetition of the primitive periodic orbit (and also for repetition numbers which are a multiple of m). The corresponding bifurcation is a period m-tupling bifurcation, which can be considered as a bifurcation of the mth traversal of the primitive orbit. The Gutzwiller approximation has to be changed only for those repetition numbers, which are close to a bifurcation. For these cases, one then has to apply uniform approximations which result in a common contribution of all orbits which take part in the bifurcation. In the present paper, a uniform approximation is derived for generic bifurcations with $m \geqslant 5$. This uniform approximation has the property that it yields the Gutzwiller contributions when one moves away from the bifurcation.

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Appendix A. The normal form for the generating function

The motion in the vicinity of a periodic orbit can be described in a reduced system. In this system the coordinate z along the orbit is the new time variable and the reduced Hamiltonian is given by the function $F = -p_z$ whose functional form is defined implicitly by $H(p_z, p_y, z, y) = E$ [20]. From the equations of motion of the original system it follows that

$$\frac{\mathrm{d}y}{\mathrm{d}z} = \frac{\partial F}{\partial p_y} \qquad \frac{\mathrm{d}p_y}{\mathrm{d}z} = -\frac{\partial F}{\partial y} \qquad \frac{\mathrm{d}F}{\mathrm{d}z} = \frac{\partial F}{\partial z}. \tag{A1}$$

In this way the original two-dimensional autonomous system is reduced in the vicinity of a periodic orbit to a one-dimensional time-dependent system with periodic time dependence. In the reduced system the energy E is only a parameter.

The system can be further simplified by a canonical transformation to the normal form of the reduced Hamiltonian system. This is done by expanding the reduced Hamiltonian around the origin in a combined Taylor expansion in the variables y and p_y and Fourier expansion in the periodic time variable z.

By a canonical transformation of the coordinates of the reduced system it is then possible to remove most of the terms in this combined expansion. An exception to this rule are terms which meet a resonance condition that is dependent on the stability of the orbit. The method is described in detail in the book of Ozorio de Almeida [8], and for that reason here we only give the result. For the case of a stable orbit close to a period m-tupling bifurcation with $m \geqslant 5$ one obtains for the expansion up to order I^{m-2}:

$$F = \tilde{\varepsilon} I + \sum_{\nu=2}^{m-2} \tilde{c}_\nu I^\nu + \sum_{\nu=0}^{[\frac{m}{2}]-2} \tilde{d}_\nu I^{\frac{m}{2}+\nu} \sin(m\Phi) + \sum_{\nu=0}^{[\frac{m}{2}]-2} \tilde{e}_\nu I^{\frac{m}{2}+\nu} \cos(m\Phi) \qquad \text{(A2)}$$

where I and Φ are canonical polar coordinates. The transformation to the normal form is done in several steps and the canonical polar coordinates are introduced in the last step $P_Y = \sqrt{2I}\cos\Phi$, $Y = \sqrt{2I}\sin\Phi$ where Y and P_Y are unbounded canonical coordinates. As can be seen from (A2) the Hamiltonian no longer depends on z.

The generating function for the transformation from the initial point of a trajectory (I, Φ) at z to a final point (I', Φ') at z' has a very similar form:

$$S(I, \Phi', E) = I\Phi' - \varepsilon I - \sum_{\nu=2}^{m-2} c_\nu I^\nu - \sum_{\nu=0}^{[\frac{m}{2}]-2} d_\nu I^{\frac{m}{2}+\nu} \sin(m\Phi') - \sum_{\nu=0}^{[\frac{m}{2}]-2} \mathrm{e}_\nu I^{\frac{m}{2}+\nu} \cos(m\Phi'). \qquad \text{(A3)}$$

This can be seen, for example, by integrating iteratively the equations of motion for I and Φ (up to the relevant order in I). By a further canonical transformation which consists of an I-dependent shift of the angle coordinate Φ, the cosine-dependent terms can be removed and one obtains

$$S(I, \Phi, E) = I\Phi - \varepsilon I - \sum_{\nu=2}^{m-2} c_\nu I^\nu - \sum_{\nu=0}^{[\frac{m}{2}]-2} a_\nu I^{\frac{m}{2}+\nu} \sin(m\Phi) \qquad \text{(A4)}$$

where $\varepsilon = m\alpha_{0,p} - 2\pi n$ when $z' = z + ml_\gamma$.

We end this section with a discussion of the kind of canonical transformations that are involved in changing the representation from (y, p_y)-coordinates to (Φ, I)-coordinates. In the reduced system the transformation is a time-dependent canonical transformation which changes the energy $-p_z$ of the reduced system. In the original system, the transformation to normal form coordinates $(z, p_z, y, p_y) \rightarrow (Z, P_z, \Phi, I)$ is an energy-dependent canonical transformation. It does not change the z-variable $(Z = z)$ but it changes the conjugate momentum p_z such that the new momentum P_z is a constant of motion if one neglects higher-order corrections to the form (A2). Furthermore, since the generating function for the transformation depends on the energy E, in general it changes the time along a trajectory. (The time along a periodic orbit is, however, not changed.) The results of Miller [14] on the form-invariance of semiclassical approximations under canonical transformations are derived for the class of autonomous transformations. They can be generalized to the present case by applying them to an extended phase space, in which energy and time are further canonical variables. This extended phase space is discussed in [15].

Appendix B. Integration over the angle variable

In this section the integral over the angle variable in (34) is performed. One proceeds in the following way. First, the integral is simplified by a substitution of the angle variable

$$\Phi' = \Theta' - \frac{b_0''}{ma_0''} \hat{\varepsilon}^{\frac{m}{2}-2} \cos(m\Theta') \qquad \text{(B1)}$$

which removes the quadratic cosine term in the exponent:

$$d_\gamma(E) \approx \mathrm{Re} \int_0^{2\pi} \mathrm{d}\Theta' \, \frac{\frac{\mathrm{d}\Phi'}{\mathrm{d}\Theta'}}{\sqrt{2(\pi\hbar)^3}} \frac{1}{m} \frac{\partial \hat{S}_\gamma}{\partial E} \frac{|\frac{\partial^2 S}{\partial I \partial \Phi'}|^{\frac{1}{2}}}{|\frac{\partial^2 S}{\partial I^2}|^{\frac{1}{2}}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\bar{S}(E) + \frac{\mathrm{i}}{\hbar}\Delta S(E)\sin(m\Theta') - \frac{\mathrm{i}\pi}{2}\nu + \frac{\mathrm{i}\pi}{4}\beta}$$
$$\times \left[\Theta(\hat{\varepsilon}) + \sqrt{\frac{\mathrm{i}\beta}{2\pi}} \, \mathrm{sign}(\hat{\varepsilon}) \int_{\Lambda'}^{\infty} \mathrm{d}X' \, \frac{1}{X'^2} \mathrm{e}^{\frac{\mathrm{i}\beta}{2}X'^2} \right] + d_0(E). \tag{B2}$$

Then the exponential prefactor is expanded in powers of $\hat{\varepsilon}$ up to order $\hat{\varepsilon}^{\frac{m}{2}-2}$ as has been discussed in section 3. In this approximation the only combination which is angle-dependent is

$$\frac{\mathrm{d}\Phi'}{\mathrm{d}\Theta'} \left| \frac{\partial^2 S}{\partial I^2} \right|^{-\frac{1}{2}} \approx B + C \sin(m\Theta') \tag{B3}$$

where B and C are given by expansions in powers of $\hat{\varepsilon}$. Both constants can be expressed in terms of classical quantities by noting that

$$\mp m^2 \Delta S(E) = \left[\frac{\mathrm{d}^2(S(I^*, \Phi', E) - I^* \Phi')}{\mathrm{d}\Theta'^2} \right]_{\sin(m\Theta') = \pm 1}$$
$$= \left[\frac{\partial^2 S}{\partial \Phi'^2} \left(\frac{\mathrm{d}\Phi'}{\mathrm{d}\Theta'} \right)^2 \right]_{I = I^*, \sin(m\Theta') = \pm 1}. \tag{B4}$$

This relation and the second line of (29) can be used to relate B and C to $\Delta S(E)$, $\mathrm{Tr}\, M_1$ and $\mathrm{Tr}\, M_2$:

$$|\,\mathrm{Tr}\, M_{1,2} - 2| = \left| \frac{\partial^2 S}{\partial I^2} \left(\frac{\mathrm{d}\Theta'}{\mathrm{d}\Phi'} \right)^2 m^2 \Delta S(E) \right|_{I = I^*, \sin(m\Theta') = \pm 1} = \frac{m^2 |\Delta S(E)|}{(B \pm C)^2} \tag{B5}$$

which determines the values of B and C. The integral expression for $d_\gamma(E)$ then has the following form:

$$d_\gamma(E) \approx \mathrm{Re} \int_0^{2\pi} \mathrm{d}\Theta' \sqrt{\frac{|\Delta S(E)|}{2(\pi\hbar)^3}} [\bar{A} + \Delta A \sin(m\Theta')] \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\bar{S}(E) + \frac{\mathrm{i}}{\hbar}\Delta S(E)\sin(m\Theta') - \frac{\mathrm{i}\pi}{2}\nu + \frac{\mathrm{i}\pi}{4}\beta}$$
$$\times \left[\Theta(\hat{\varepsilon}) + \sqrt{\frac{\mathrm{i}\beta}{2\pi}} \, \mathrm{sign}(\hat{\varepsilon}) \int_{\Lambda'}^{\infty} \mathrm{d}X' \, \frac{1}{X'^2} \mathrm{e}^{\frac{\mathrm{i}\beta}{2}X'^2} \right] + d_0(E) \tag{B6}$$

where $\bar{A}$ and ΔA are defined as

$$\begin{aligned} \bar{A} &= \frac{1}{2} \left[\frac{T_1(E)}{\sqrt{|\,\mathrm{Tr}\, M_1 - 2|}} + \frac{T_2(E)}{\sqrt{|\,\mathrm{Tr}\, M_2 - 2|}} \right] \\ \Delta A &= \frac{1}{2} \left[\frac{T_1(E)}{\sqrt{|\,\mathrm{Tr}\, M_1 - 2|}} - \frac{T_2(E)}{\sqrt{|\,\mathrm{Tr}\, M_2 - 2|}} \right]. \end{aligned} \tag{B7}$$

In the present approximation $T_1(E) = T_2(E)$ and the definition of $\bar{A}$ and ΔA in terms of these periods is somewhat ambiguous. It can be justified by a higher-order calculation as is done in appendix C.

In a last step before performing the integral over Θ' the angular dependence of the boundary Λ' of the X'-integral

$$\Lambda' = \sqrt{\frac{2}{\beta\hbar}(I^* \Phi' - S(I^*, \Phi', E))} = \sqrt{\frac{2}{\beta\hbar}(S_0(E) - \bar{S}(E) - \Delta S(E)\sin(m\Theta'))}$$

is removed by the substitution $X = \sqrt{X'^2 + \frac{2}{\beta\hbar}\Delta S(E)\sin(m\Theta')]}$. Within the present approximation for the exponential prefactor one obtains

$$\int_{\Lambda'}^{\infty} \mathrm{d}X' \, \frac{1}{X'^2} \mathrm{e}^{\frac{\mathrm{i}\beta}{2}X'^2} \approx \int_{\Lambda}^{\infty} \mathrm{d}X \, \frac{1}{X^2} \left[1 + \frac{3}{\beta\hbar X^2}\Delta S(E)\sin(m\Theta')\right] \mathrm{e}^{\frac{\mathrm{i}\beta}{2}X^2 - \frac{\mathrm{i}}{\hbar}\Delta S(E)\sin(m\Theta')} \tag{B8}$$

where $\Lambda = \sqrt{\frac{2}{\beta\hbar}[S_0(E) - \bar{S}(E)]}$. The integral expression (B8) is inserted into (B6) and results in

$$\begin{aligned} d_\gamma(E) \approx{}& \Theta(\hat{\varepsilon}) \, \mathrm{Re} \sqrt{\frac{\Delta S(E)}{2(\pi\hbar)^3}} \int_0^{2\pi} \mathrm{d}\Theta' \, [\bar{A} + \Delta A \sin(m\Theta')] \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\bar{S}(E) + \frac{\mathrm{i}}{\hbar}\Delta S(E)\sin(m\Theta') - \frac{\mathrm{i}\pi}{2}\nu + \frac{i\pi}{4}\beta} \\ & + \mathrm{sign}(\hat{\varepsilon}) \, \mathrm{Re} \sqrt{\frac{\mathrm{i}\beta|\Delta S(E)|}{4\pi^4\hbar^3}} \int_{\Lambda}^{\infty} \mathrm{d}X \, \frac{1}{X^2} \\ & \times \int_0^{2\pi} \mathrm{d}\Theta' \left[\bar{A} + \Delta A \sin(m\Theta') + \frac{3\bar{A}}{\beta\hbar X^2}\Delta S(E)\sin(m\Theta')\right] \\ & \times \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\bar{S}(E) + \frac{\mathrm{i}\beta}{2}X^2 - \frac{\mathrm{i}\pi}{2}\nu + \frac{\mathrm{i}\pi}{4}\beta} + d_0(E) \end{aligned} \tag{B9}$$

which is now in a form in which the integrations can be performed. It results in the final formula (37).

Appendix C. Calculation in higher order for $m \geqslant 7$

In this section, the derivation of formula (37) is carried out for $m \geqslant 7$ by increasing the order of the expansion in normal form coordinates by one. Then the periods of the two satellite orbits are different if the system is generic. As will be shown, the final result (37) will be the same.

The expansion of the generating function $S(I, \Phi', E)$ up to order I^{m-1} is given by

$$S(I, \Phi', E) = I\Phi' - \varepsilon I - \sum_{\nu=2}^{m-1} c_\nu I^\nu - \sum_{\nu=0}^{[\frac{m}{2}]-1} a_\nu I^{\frac{m}{2}+\nu} \sin(m\Phi'). \tag{C1}$$

The stationary points of the integral over Φ' are again determined by $\cos(m\Phi') = 0$, and the stationary point $I^*(\Phi')$ of the integral over I can be expanded in powers of $\hat{\varepsilon}$ up to order $\hat{\varepsilon}^{m-2}$:

$$I^*(\Phi') = \sum_{\nu=2}^{m-1} c'_\nu \hat{\varepsilon}^{\nu-1} + \sum_{\nu=0}^{[\frac{m}{2}]-1} a'_\nu \hat{\varepsilon}^{\frac{m}{2}+\nu-1} \sin(m\Phi') + \sum_{\nu=0}^{1} b'_\nu \hat{\varepsilon}^{m+\nu-3} \cos^2(m\Phi'). \tag{C2}$$

In the cases $m = 5$ and $m = 6$ there is an additional term of order $\hat{\varepsilon}^{\frac{3m}{2}-5}$, and for $m = 5$ a further term of order $\hat{\varepsilon}^{2m-7}$.

Inserting the expansion for $I^*(\Phi')$ into the expression (C1) for the generating function $S(I, \Phi', E)$ yields

$$S(I^*, \Phi', E) = I^*\Phi' + \sum_{\nu=2}^{m-1} c''_\nu \hat{\varepsilon}^\nu + \sum_{\nu=0}^{[\frac{m}{2}]-1} a''_\nu \hat{\varepsilon}^{\frac{m}{2}+\nu} \sin(m\Phi') + \sum_{\nu=0}^{1} b''_\nu \hat{\varepsilon}^{m+\nu-2} \cos^2(m\Phi') \tag{C3}$$

and from this the expansions of the actions $S_1(E)$ and $S_2(E)$ of the satellite orbits are obtained. They are determined by

$$\bar{S}(E) = \frac{S_1(E) + S_2(E)}{2} = S_0(E) + \sum_{\nu=2}^{m-1} c''_\nu \hat{\varepsilon}^\nu$$
$$\Delta S(E) = \frac{S_1(E) - S_2(E)}{2} = \sum_{\nu=0}^{[\frac{m}{2}]-1} a''_\nu \hat{\varepsilon}^{\frac{m}{2}+\nu}. \tag{C4}$$

The following calculations are done as in the main section and in appendix B with the difference that all quantities in the exponent are now expanded up to order $\hat{\varepsilon}^{m-1}$ and all quantities in the exponential prefactor up to order $\hat{\varepsilon}^{\frac{m}{2}-1}$. The exponent in the integral expression (34) for $d_\gamma(E)$ then contains two terms with a quadratic cosine-dependence on the angle, and these terms are removed by a substitution of the angle variable of the form

$$\Phi' = \Theta' + \sum_{\nu=0}^{1} d_\nu \hat{\varepsilon}^{\frac{m}{2}-2+\nu} \cos(m\Theta'). \tag{C5}$$

Again one has

$$\frac{\mathrm{d}\Phi'}{\mathrm{d}\Theta'} \left| \frac{\partial^2 S}{\partial I^2} \right|^{-\frac{1}{2}} \approx B + C \sin(m\Theta') \tag{C6}$$

where B and C are determined by (B5), but now also $\partial S_\gamma/\partial E$ and $\partial^2 S/\partial I \partial \Phi'$ depend on Θ'. An expansion of the whole exponential prefactor in (B2) up to order $\hat{\varepsilon}^{\frac{m}{2}-1}$ results in

$$d_\gamma(E) \approx d_0(E) + \mathrm{Re} \int_0^{2\pi} \mathrm{d}\Theta' \sqrt{\frac{|\Delta S(E)|}{2(\pi\hbar)^3}} [\bar{A} + \Delta A \sin(m\Theta') + D \cos(m\Theta')]$$
$$\times \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\bar{S}(E) + \frac{\mathrm{i}}{\hbar}\Delta S(E)\sin(m\Theta') - \frac{\mathrm{i}\pi}{2}\nu + \frac{\mathrm{i}\pi}{4}\beta} \left[\Theta(\hat{\varepsilon}) + \sqrt{\frac{\mathrm{i}\beta}{2\pi}} \,\mathrm{sign}(\hat{\varepsilon}) \int_{\Lambda'}^{\infty} \mathrm{d}X' \, \frac{1}{X'^2} \mathrm{e}^{\frac{\mathrm{i}\beta}{2}X'^2} \right] \tag{C7}$$

where $D = \bar{A} a_0^{(5)} \hat{\varepsilon}^{\frac{m}{2}-1}/2$ and $\bar{A}$ and ΔA are defined as before, but now with an unambiguous dependence on $T_1(E)$ and $T_2(E)$.

The expression (B8) for the substitution from X' to X is not modified in the present approximation, and one obtains the following result for $d_\gamma(E)$:

$$d_\gamma(E) \approx d_0(E) + \Theta(\hat{\varepsilon}) \,\mathrm{Re} \sqrt{\frac{\Delta S(E)}{2(\pi\hbar)^3}} \int_0^{2\pi} \mathrm{d}\Theta' \, [\bar{A} + \Delta A \sin(m\Theta') + D \cos(m\Theta')]$$
$$\times \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\bar{S}(E) + \frac{\mathrm{i}}{\hbar}\Delta S(E)\sin(m\Theta') - \frac{\mathrm{i}\pi}{2}\nu + \frac{\mathrm{i}\pi}{4}\beta} + \mathrm{sign}(\hat{\varepsilon}) \,\mathrm{Re} \sqrt{\frac{\mathrm{i}\beta|\Delta S(E)|}{4\pi^4\hbar^3}} \int_{\Lambda}^{\infty} \mathrm{d}X \, \frac{1}{X^2}$$
$$\times \int_0^{2\pi} \mathrm{d}\Theta' \left[\bar{A} + \Delta A \sin(m\Theta') + D \cos(m\Theta') + \frac{3\bar{A}}{\beta\hbar X^2} \Delta S(E) \sin(m\Theta') \right]$$
$$\times \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\bar{S}(E) + \frac{\mathrm{i}\beta}{2}X^2 - \frac{\mathrm{i}\pi}{2}\nu + \frac{\mathrm{i}\pi}{4}\beta}. \tag{C8}$$

The integral over Θ' yields the same result (37) as in the main section since the integral over the additional terms in the integrand of (C8) vanishes.

Appendix D. Billiard systems

The semiclassical contribution of a stable orbit in a billiard system can differ from the general form in (1). It is given by

$$d_0(E) = \frac{1}{\pi\hbar}\frac{T_0(E)}{m}\frac{\cos(\frac{S_0(E)}{\hbar} - \frac{\pi\nu_0}{2})}{2|\sin(\frac{\alpha_0}{2})|} \tag{D1}$$

where the angle α_0 contains a contribution of π from every reflection on a billiard wall and the index ν_0 is

$$\nu_0 = 2\left[\frac{\alpha_0}{2\pi}\right] + 1 \pm n_0 \tag{D2}$$

where the brackets denote the integer part and n_0 is the number of reflections. Here and in the following upper and lower signs correspond to Dirichlet and Neumann boundary conditions, respectively.

It is convenient to define a new angle $\hat{\alpha}_0 = \alpha_0 - n_0\pi$ by subtracting the boundary contributions from α_0. Then the expression for $d_0(E)$ depends on whether n_0 is even or odd

$$d_0(E) = \begin{cases} \frac{1}{\pi\hbar}\frac{T_0(E)}{m}\frac{\sin(\frac{S_0(E)}{\hbar})}{2\sin(\frac{\hat{\alpha}_0}{2})} & n_0 \text{ even} \\ \mp\frac{1}{\pi\hbar}\frac{T_0(E)}{m}\frac{\cos(\frac{S_0(E)}{\hbar})}{2\cos(\frac{\hat{\alpha}_0}{2})} & n_0 \text{ odd.} \end{cases} \tag{D3}$$

In the case of even n_0, the calculations are done exactly as in the main section, and all formulae and results are the same when α_0 is replaced by $\hat{\alpha}_0$.

In the case of odd n_0, the formulae have to be slightly modified: a bifurcation occurs when $\hat{\alpha}_0 = (2n + 1)\pi$ for integer n, and ε is defined as $\varepsilon = \hat{\alpha}_0 - (2n + 1)\pi$. The index ν is then given by $\nu = 2n \mp 1$. The final result has the same form as in (37), but $d_0(E)$ is now given by the form in (D3) for odd n_0, the indices of the satellite orbits are $\nu_1 = 2n \mp 1 - (\beta - 1)/2$ and $\nu_2 = 2n \mp 1 - (\beta + 1)/2$, and the sine-function in the integral expression of the interference term has to be replaced by a $\pm$cosine-function.

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